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Duality cascades and duality walls

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Abstract: We recast the phenomenon of duality cascades in terms of the Cartan matrix associated to the quiver gauge theories appearing in the cascade. In this language, Seiberg dualities for the different gauge factors correspond to Weyl reflections. We argue that the UV behavior of different duality cascades depends markedly on whether the Cartan matrix is affine ADE or not. In particular, we find examples of duality cascades that can't be continued after a finite energy scale, reaching a "duality wall", in terminology due to M. Strassler. For these duality cascades we suggest the existence of a UV completion in terms of a little string theory.

Keywords: Duality in Gauge Field Theories, Supersymmetry and Duality.

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1. Introduction

One of the most interesting developments in the AdS/CFT correspondence [1] has been its extension to non-conformal field theories. It has been appreciated for some time that one way to break conformal symmetry is to consider, on the string side, branes that wrap vanishing cycles. In particular, in a beautiful series of works [2]–[7] the gauge theories that arise when one places regular D3 branes plus wrapped D5 branes at the simplest conifold singularity have been studied, and compared with the corresponding supergravity duals (see refs. [8, 9] for reviews).

In particular, in reference [6] Klebanov and Strassler (KS), managed to write a regular IIB supergravity solution corresponding to the case of having N regular and M fractional D3 branes, whose dual description involves 4d $\mathcal{N}=1$ $\mathrm{SU}(N+M)\times\mathrm{SU}(N)$ gauge theories. A key feature of that solution is that the 5-form flux varies in the radial direction of the AdS part of the background. This translates, in the dual gauge theory, into a running of the gauge couplings. As the couplings run, eventually one of them gets strong, and the best description is in terms of a similar gauge theory, but with lower rank $N\to N-M$. In [6], this change in ranks was interpreted as Seiberg duality [10]. In this fashion, the gauge dual of the supergravity solution is a RG flow, where the effective field theory description changes an arbitrarily large number of times, as we vary the energy scale. This phenomenon was dubbed a "duality cascade" in [6].

An important point is that in the presence of wrapped D5-branes, none of these $\mathrm{SU}(N+M)\times\mathrm{SU}(N)$ theories is asymptotically free so, strictly speaking, they are not well defined quantum field theories at all energy scales. This raises a number of interesting questions, e.g. whether the KS gravitational background is dual to an $N = \infty$ theory, rather than a finite N gauge theory, or whether one can find a UV completion of this RG flow in terms of ordinary field theory.

Given the remarkable features of this supergravity solution and its dual interpretation, it seems natural to try to generalize it to other examples. For instance, it would be desirable to understand at a more general level the relation between the asymptotic (IR in the geometry) behavior of supergravity solutions with varying flux, and the UV properties of the field theory dual. Generalizing the KS solution to other backgrounds/brane configurations does not seem straightforward, but at least some questions can be addressed without knowing the supergravity solution. Some steps along that direction were taken in [11], where the duality cascade of [6] was reinterpreted in terms of geometric transitions among vanishing 2-cycles. This identification allowed for immediate generalizations, and duality cascades corresponding to transverse D3 branes plus wrapped D5-branes on ADE fibrations (including ADE conifolds) were also discussed in [11] (see also [12]).

It has been known since the original work of D-branes on ADE orbifolds [13, 14], that D-branes at orbifolds have worldvolume theories with matter content dictated by the corresponding ADE Dynkin diagram, and in particular, adding D-branes transverse to the orbifold corresponds to adding an affine node to the Dynkin diagram. This is also true for ADE conifolds, so placing D3-branes transverse to a conifold yields affine ADE gauge theories, whose Weyl group is infinite. It was pointed out in [11] that this infinite order of the affine Weyl group is the reason underlying the existence of duality cascades.

In this paper we further explore possible duality cascades, considering general $\mathcal{N} = 1$ quiver theories, beyond the ADE or affine ADE cases, from a purely field theoretical point of view. We recast the phenomenon of duality cascades in terms of the Cartan matrix one can naturally associate to a quiver gauge theory — which is $\hat{A}_1$ in the KS example —, and we argue that the UV behavior of the duality cascade depends markedly of the type of Cartan matrix. In this language, Seiberg dualities are realized as Weyl reflections [11].

The main novelty of the present work is a discussion of duality cascades involving quiver gauge theories whose Cartan matrix has exactly one negative eigenvalue. In particular we analyze in detail the simplest example: $\mathcal{N} = 1$ gauge theories with gauge group $\mathrm{SU}(N_1) \times \mathrm{SU}(N_2)$ with $k > 2$ chirals in $(N_1, \bar{N}_2)$ and k chirals in $(N_2, \bar{N}_1)$. As we consider the different flows that arise as we vary N_1 and N_2, some surprises arise, as all the flows seems to converge towards the UV, and furthermore, the scales at which we are supposed to change the effective description pile up at an accumulation point. This behavior had been already observed in some models by M. Strassler, who named it a "duality wall" [15]. These properties lead us to speculate that for each fixed k, a new kind of theory exists (perhaps some little string theory), controlling the UV of all these flows.

The organization of the paper is the following. In section 2 we define the Cartan matrix of a quiver gauge theory, and relate its Weyl reflections with Seiberg duality applied to different gauge factors of the quiver theory. Duality cascades arise as flows connecting theories that are roots of the same Weyl orbit. In section 3 we review the cases of quiver gauge theories with ADE and affine ADE Cartan matrices. These have been extensively studied in the literature, as they arise as the worldvolume theory of D-branes in different singularities. Then we move in section 4 to consider quiver gauge theories with a hyperbolic Cartan matrix, and discuss in detail a simple example, stressing the qualitative difference in behavior with the KS duality cascade. Section 5 contains conclusions and ideas for further work.

2. Weyl reflections and Seiberg dualities

We will consider $\mathcal{N} = 1$ gauge theories of the quiver type: the gauge group is $G = \prod_{i=1}^{n} \mathrm{SU}(N_i)$, and chiral superfields transform in bifundamental representations $(N_i, \bar{N}_j)$. In the context of string theory, quiver gauge theories arise as the worldvolume theories of D-branes [14, 16]. In what follows, we apply to quiver gauge theories some ideas familiar from the study of Lie algebras, starting with the definition of the Cartan matrix associated with a quiver gauge theory.[1] As we will see, the Cartan matrix encodes the matter content of the theory, while theories that differ by the ranks of the gauge groups can be thought of as vectors on a root space. In particular, Weyl reflections change the ranks of the gauge groups as Seiberg dualities do [11].

Our starting point is the *Cartan matrix* of a quiver gauge theory with gauge group $G = \prod_{i=1}^{n} \mathrm{SU}(N_i)$, $i = 1, \ldots, n$, defined as

$$C_{ij} = 2\delta_{ij} - \#\text{ of chirals in } (N_i, \bar{N}_j)\,. \tag{2.1}$$

Note that the Cartan matrix thus defined is insensitive to the possible existence of a superpotential, or to the presence of matter in representations other than the bifundamental. In the present paper we restrict to $\mathcal{N} = 1$ gauge theories with chiral matter in $\mathcal{N} = 2$ hypermultiplets, namely for every chiral in $(N_i, \bar{N}_j)$, there is a chiral in $(\bar{N}_i, N_j)$. This means the Cartan matrix just defined is symmetric for these theories.

Once we specify the superpotential, a particular gauge theory is determined by the ranks of the gauge groups, $(N_1, \ldots, N_n)$. We can think of this n-tuplet of positive integers as a *positive root* of an algebra associated with the generalized Cartan matrix (2.1). For each gauge group $\mathrm{U}(N_i)$, the number of colors and flavors, in the limit where all the other gauge couplings are weak, is given by

$$N_c(i) = N_i \qquad N_f(i) = \sum_{j \neq i} |C_{ij}| N_j\,.$$

To define the Weyl reflections, first we associate a simple root $\vec{\alpha}_i$, to each node of the diagram. Then, for each simple root $\vec{\alpha}_i$ we define an action over an arbitrary root $\vec{\beta}$ of the algebra (in the remaining, there is no summation over i)

$$R_i(\vec{\beta}) = \vec{\beta} - (\vec{\beta} \cdot \vec{\alpha}_i)\vec{\alpha}_i\,.$$

[1] See [17] and the appendix of [18] for introductions to quiver theory and some of its applications to supersymmetric gauge theories. The Cartan matrix associated with a quiver gauge theory has appeared, in a more or less explicit way, in a number of works [19, 17, 18]. As far as I am aware, the associated Weyl reflections were first discussed in the physics literature in [18]; their relation to Seiberg dualities was pointed out — in the ADE and affine ADE cases — in [11], although it is implicit in previous work realizing Seiberg duality through D-branes on Calabi-Yau's [20]–[24].

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In particular, acting on another simple root,

$$R_i(\vec{\alpha}_j) = (\delta_{jk} - \delta_{ik} C_{ij})\vec{\alpha}_k \equiv R^{(i)}_{jk}\vec{\alpha}_k \, .$$

It is easy to see that $R^2_{(i)} = 1$, and that $R_{(i)}C = CR^T_{(i)}$. Thinking of our gauge theory as a vector $\vec{N} = N_i \vec{\alpha}_i$, in the "root space" with components $(N_1, \ldots, N_n)$, we are actually more interested in taking the passive point of view on the reflection, and consider its effect on the components of the vector. By requiring $\vec{N} = \vec{N}'$, we deduce that under the i-th Weyl reflection,

$$N'_j = N_k R^{(i)}_{kj} = (\delta_{jk} - \delta_{ij} C_{ik}) N_k \, , \tag{2.2}$$

where we have used that $R^2_{(i)} = 1$. Note that the only N that changes under R_i is N_i. Also, the length square of the vector, $\vec{N}^2 = C_{ij} N_i N_j$, is invariant under Weyl reflections, something that we will be using repeatedly in what follows. In physical terms, a Weyl reflection sends a theory with gauge group $G = \prod_{i=1}^n \mathrm{SU}(N_i)$ to another with gauge group $G = \prod_{i=1}^n \mathrm{SU}(N'_i)$ and a chiral in $(N'_j, \bar{N}'_k)$ for every original chiral in $(N_j, \bar{N}_k)$. It is revealing to write the action of the i-th Weyl reflection on $N_c(i), N_f(i)$,

$$N_c(i) \to N_f(i) - N_c(i) \qquad N_f(i) \to N_f(i) \, .$$

We see that the effect on $N_c(i), N_f(i)$ is identical to that of a Seiberg duality applied on the $i-$th gauge group, which is also an operation that squares to the identity. For general quiver theories, it would be wrong to conclude that Weyl reflections correspond to Seiberg duality: to start with, the definition of Weyl reflections doesn't depend at all on possible superpotentials; more importantly, in the original Seiberg duality, the dual magnetic theory has a massless meson, and a superpotential. Therefore, Weyl reflections have a chance to correspond to Seiberg duality only for theories where that massless meson doesn't appear in the dual theory.

It is well known [25, 26] that particular choices of the superpotential give mass to that meson. For a $\mathrm{SU}(N)$ theory with fundamental and antifundamental matter, a quartic superpotential does the job. Furthermore, it is self-dual, in the sense that the magnetic theory presents a similar superpotential in the dual quarks. Luckily, for $\mathcal{N} = 1$ quiver theories, these quartic superpotentials appear quite naturally. If we start with a $\mathcal{N} = 2$ quiver theory, it has a superpotential

$$W_{\mathcal{N}=2} = \sum{}' Q^I_{ij}(\phi_j) Q^I_{ji} \, ,$$

where $\sum'$ means the sum is weighted with the appropiate signs. We can deform this theory to a $\mathcal{N} = 1$ theory by adding some superpotentials for the adjoint fields

$$W_{\mathcal{N}=1} = W_{\mathcal{N}=2} + \sum W_i(\phi_i) \, .$$

If at some energy scale we integrate out the ϕ's, we indeed obtain quartic superpotentials for the Q's. In this paper we will be considering examples with such quartic superpotentials. It would be interesting to explore whether there are other superpotentials that are self-dual and get rid of the massless meson.

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For $\mathcal{N} = 1$ quiver theories, such a quartic superpotential appears, for instance, when one considers the worldvolume theory of N D3 branes transverse to a conifold, as discussed by Klebanov and Witten [2], and its self-duality plays a pivotal role in the duality cascade — to be reviewed in section 3.2 — that arises when one considers additional wrapped D5 branes [6]. Finally, the dualization of a gauge factor in $\mathcal{N} = 1$ quiver theories was analyzed in [11], where the Weyl reflected theory was obtained after duality plus higgsing, following the analysis presented in the appendix of [27].

2.1 Weyl groups and RG flows

For a given quiver gauge theory, the set of Weyl reflections (2.2) forms a group — the Weyl group — and in principle, by judiciously acting with these reflections on a theory, it can be mapped to infinitely many others in the same Weyl orbit. The question arises of whether these theories are related dynamically. An example in the literature where infinitely many $\mathcal{N} = 1$ quiver gauge theories are related by Seiberg duality is the duality cascade of [6]. In [11], this duality cascade was recasted in the language of Weyl reflections, and some generalizations were presented, although no attempt was made to write down the corresponding supergravity solutions. Our aim is to consider further generalizations of the original duality cascade; we will argue that many features of the flow are neatly encoded in the Cartan matrix (2.1) of the theories entering the cascade, justifying the introduction of this formalism.

For general quiver theories, our approach is the following: once the matter content — i.e., the Cartan matrix — and the superpotential are fixed, the starting point is a particular gauge theory — i.e., a n-tuplet $(N_1, \ldots, N_i, \ldots, N_n)$ — , an energy scale μ and some values for the gauge couplings at that scale $g_i(\mu)$. Requiring that the gauge couplings are positive fixes whether the scales Λ_i are bigger or smaller than μ. Then we flow towards the IR, by changing μ. Eventually one of the gauge couplings will become strong, and at that energy scale we perform Seiberg duality — i.e, a Weyl reflection — on the particular gauge factor. That yields a new theory $(N_1, \ldots, N_i', \ldots, N_n)$ with only the N of the dualized factor changed. As we flow to the IR, it makes sense to dualize as long as the electric coupling is stronger than the magnetic one in the IR. For $\mathrm{SU}(N_c)$ with N_f flavors, this is the case as long as $N_f < 2N_c$, i.e. $N_c' < N_c$ [10]. For the quiver theories, we also have a superpotential and the other gauge factors, but these shouldn't change the fact that as we flow to the IR, $N_c' < N_c$, i.e. the ranks of the gauge groups keep decreasing.

Although much attention has been devoted to the IR properties of the KS duality cascade, its UV behavior also raises a number of interesting questions. Since at each energy scale the effective gauge theory is not asymptotically free — as the two beta functions have different sign — it is not immediate to find a UV completion of that RG flow, purely in terms of ordinary field theory.

For the KS cascade, the supergravity solution provides the UV behavior; on the other hand, for some of the examples we will consider later, we lack a gravity/string dual description of any flow. Still, we would like to ask what is the UV behavior of these cascades. Clearly, given an effective field theory, there is no unique way to flow "upwards", towards the UV. What we propose is to reconstruct a particular flow, by mimicking the behavior of the KS flow, i.e, let the couplings run towards the UV and every time a particular coupling gets strong, we perform a Weyl reflection. That changes the sign of the corresponding beta function, so that coupling now gets weaker towards the UV, allowing us to let the couplings run upstream until another coupling gets strong, at which point we repeat the same operation. A point of caution is required: for specific examples, dynamically we will see that this prescription calls for applying Seiberg duality on a particular gauge factor when the rest of the couplings are not weak. We will comment below on to what extend this procedure is justified.

Finally, let's mention that some examples of duality cascades for quiver gauge theories with non-symmetric Cartan matrix were considered in [11]. In the cases studied there, the matter content changed at each step of the cascade, so the theories appearing in a particular flow don't have a common Cartan matrix. Physically, this corresponds to the fact that in these cases some of the Seiberg mesons remain massless (as can be anticipated by demanding that the dual theory has no anomalies). The UV behavior of such cascades is an open question left for future work.

3. UV behavior and Cartan matrices

The classification of Cartan matrices according to their eigenvalues is well established, and we will argue that it dictates the basic features of duality cascades. The simply-laced positive definite ones follow the ADE classification. In string theory, $\mathcal{N}=1$ or $\mathcal{N}=2$ ADE and affine ADE theories can appear as worldvolume theories of D-branes [14, 16]. From a purely field theoretical point of view, it seems worth considering more general quiver gauge theories. In particular, we will study quiver gauge theories with a hyperbolic Cartan matrix, for which we are not aware of any realization in string theory.

3.1 ADE theories

The classification of positive definite Cartan matrices is familiar from the study of simple Lie algebras. We pay special attention to the ones corresponding to simply-laced algebras, namely the ADE matrices. The Weyl group is finite, so just a finite number of gauge theories are related by Weyl reflections in this case.

For $\mathcal{N}=2$, these theories can be obtained in type II string theory by placing fractional branes on a $\mathbb{C}^2/\Gamma$ orbifold, if we allow only fractional but no regular branes. Recent work on finding the supergravity duals for these theories is nicely summarized in [28]. We can break these $\mathcal{N}=2$ theories to $\mathcal{N}=1$ by adding some superpotential for the adjoint fields. In the case of quadratic superpotentials $W(\phi_i)\sim\phi_i^2$, these $\mathcal{N}=1$ theories appear as worldvolume theories of fractional branes in the ADE conifolds considered by [29]. More general superpotential deformations appear by considering branes in other ADE fibrations [11].

3.2 Affine ADE theories and duality cascades

Consider a quiver theory whose Cartan matrix is an affine ADE matrix, i.e. one of the eigenvalues is zero and the rest are positive. If we denote by d_i the Dynkin indices of the affine Dynkin diagram, the eigenvector with zero eigenvalue is $(Nd_0, Nd_1, \ldots, Nd_r)$. We have to differenciate between gauge theories corresponding to the zero eigenvector (Nd_i) and the rest.

i) In all the cases considered so far in the literature, the theories with $G = \prod_{i=1}^{n} \mathrm{U}(Nd_i)$ flow to a non-trivial IR fixed point. In the context of string theory, these theories appear as the worldvolume of N regular branes placed transverse to a $\mathbb{C}^2/\Gamma$ singularity (for $\mathcal{N}=2$) or an ADE fibration (for $\mathcal{N}=1$).

For $\mathcal{N}=2$ theories, the β functions are exactly zero. These theories are discussed in [19, 30]. For $\mathcal{N}=1$ theories, the $\hat{A}_1$ case was considered by Klebanov and Witten [2], who studied the worldvolume theory of N D3 branes transverse to the A_1 conifold. The gauge theory they obtained is $\mathrm{U}(N)\times\mathrm{U}(N)$ with 2 chirals in $(N,\bar{N})$ and 2 chirals in $(\bar{N},N)$, so indeed the Cartan matrix is $\hat{A}_1$ and (N,N) is the vector with zero eigenvalue.

The generalization to $\widehat{ADE}$ theories with quadratic superpotentials $W(\phi_i) = m_i\phi^2$ was studied by [29, 31]. The theories that arise in the discussion of $\mathbb{Z}_k$ orbifolds of the conifold [32] turn out to have affine A_{2k-1} Cartan matrix. In [11] it was argued that the $\mathcal{N}=1$ theories with gauge group $G = \prod_{i=1}^{n} \mathrm{U}(Nd_i)$ and superpotentials $W(\phi_i) = \phi^{k+1}$ also present IR fixed points, for arbitrary k.

ii) The case when the gauge group is not the zero eigenvector is realized in string theory by placing both fractional and regular branes at the orbifold or ADE conifold (or ADE fibration). In this case we have an arbitrarily large number of theories related by a RG flow. To characterize the possible Weyl orbits, recall that $\vec{N}^2$ is invariant under the flow; in the affine case we have $\vec{N}^2 \geq 0$. Embedding the 'root space' $\mathbb{N}^n$ in $\mathbb{R}^n$ and allowing momentarily the $N's$ to be real, the equation $\vec{N}^2 = s^2$ describes ellipsoids times $\mathbb{R}$. The degenerate case $s^2 = 0$ corresponds to a straight line. Weyl orbits consist of points with integer coefficients in these hypersurfaces.

In the $\hat{A}_1$ case, the supergravity solutions have been written down: for $\mathcal{N}=2$, Polchinski [33] presented a (singular) supergravity solution corresponding to N D3 branes and M wrapped D5 branes on $\mathbb{C}^2/\mathbb{Z}_2$. The physics of that $\mathcal{N}=2$ RG flow is not expected to be a duality cascade, but rather a series of higgsings [34]. For $\mathcal{N}=1$, Klebanov and Strassler [6] constructed a regular IIB supergravity solution, obtained by considering N_1 D3 branes transverse to the conifold singularity, plus M D5 branes wrapping a S^2 cycle. Defining $N_2 = N_1 + M$, these theories are of the form $\mathrm{SU}(N_1)\times\mathrm{SU}(N_2)$ with 2 chirals in $(N_1,\bar{N}_2)$ and 2 chirals in $(\bar{N}_1,N_2)$ — so indeed its Cartan matrix is $\hat{A}_1$.

Let's illustrate the formalism introduced with the KS example. In this case $\vec{N}^2 = 2(N_1^2 + N_2^2 - 2N_1N_2) = 2M^2$, which tells us that $|M|$ is constant along the flow. The orbits reduce to $S^0\times\mathbb{R}$, i.e. two straight lines in the $N_1\times N_2$ plane, given by $N_2 = N_1 \pm M$. As we flow towards the IR, we reach a theory for which, if we tried to apply yet another Weyl reflection we would run into the region with negative N's. This is the IR endpoint of the cascade, and for fixed M we have M possible such fixed points, with theories $\mathrm{SU}(M+p)\times\mathrm{SU}(p)$, $p = 1,\ldots,M$. The physics depends on the ratio p/M [6] : for $p \ll M$ KS argue that there is confinement for some range of energy scales.

In figure 1 we plot these flows in the $N_1 - N_2$ plane. Using the $\mathbb{Z}_2$ symmetry between the two gauge factors, we represent theories choosing always $N_2 > N_1$.

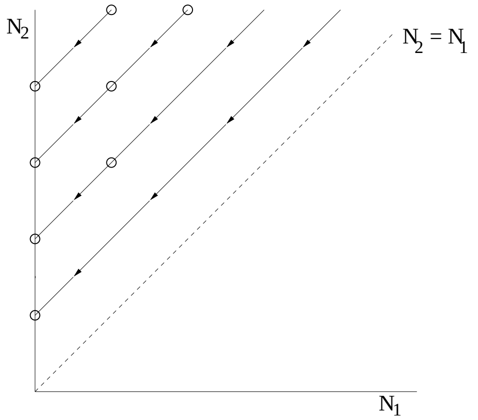

Figure 1: Affine flow of the Klebanov-Strassler duality cascade. Different flows can have the same value of $M = N_2 - N_1$; their IR endpoints are represented by the dots.

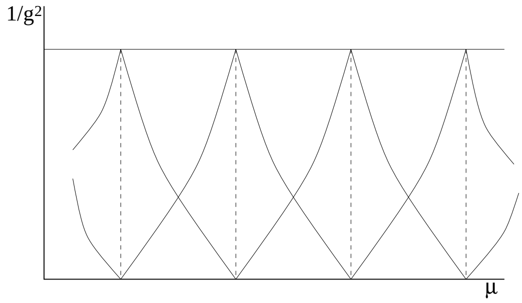

Figure 2: Periodic behavior of the two gauge couplings in the Klebanov-Strassler duality cascade.

Another feature of the KS duality cascade is that towards the UV the $N's$ grow linearly with the number of steps in the cascade, namely, after n steps, $\mathrm{SU}(N) \times \mathrm{SU}(N+M)$ has turned into $\mathrm{SU}(N+nM)\times\mathrm{SU}(N+(n+1)M)$. This "linear growth" of the N's is general for all affine ADE quiver theories, since for affine ADE — except $\hat{A}_1$ — there is at most one hypermultiplet transforming under any two gauge factors, so after n steps, if we rewrite the new N's in term of the original ones, each coefficient is going to be at most n. Finally, for future comparison, let us remember that in the KS cascade, the couplings behave periodically.

4. Hyperbolic theories and duality walls

Once we leave the realm of ADE or affine ADE Cartan matrices, there is no full classification of the possible Cartan matrices. For the sake of concreteness, we will restrict in the following to gauge theories with a hyperbolic Cartan matrix, i.e., matrices with a single negative eigenvalue and all the other positive. These bilinear forms are familiar from General Relativity, and in classifying such theories we can borrow some of the language of GR. Contrary to the previous cases, we are not aware of any realization of these theories as world-volume theories for D-branes, as this would amount to engineer configurations with an arbitrary number of hypermultiplets in a given bifundamental representation. After a few generalities, we will study in some detail the simplest example.

A first novelty, compared with the affine theories, is that the equation $\vec{N}^2 = s^2$ can now have positive, zero or negative s^2 (we could call them space-like, light-like and time-like theories). Again, allowing momentarily the N's to be real, these equations correspond to hyperboloids in $\mathbb{R}^n$. The degenerate case $\vec{N}^2 = 0$ corresponds to a cone going through the origin (the light-cone).

Applying, as in [19], the Perron-Frobenius theorem to the adjacency matrix $A = 2I - C$, we learn that the eigenvector with smallest eigenvalue of the Cartan matrix can always be chosen to have all entries positive. For a hyperbolic matrix, the smallest eigenvalue is the negative one, so we deduce that the eigenvector with the negative eigenvalue (the one giving the "time direction") can be chosen to have all entries positive, and it corresponds to a sensible gauge theory. Also, note that the axis $(0, \ldots, N_i, \ldots, 0)$ belong to the $\vec{N}^2 > 0$ region.

We want to argue that the typical IR behavior of flows with $\vec{N}^2 > 0$ and $\vec{N}^2 < 0$ is different. Before proceeding with the argument, it will be useful to introduce a couple of definitions. First, for a given vector $\vec{N}$, we define the vector $\Delta\vec{N}$ with components $\Delta N_i = (R^{(i)}(\vec{N}))_i - N_i = N_i' - N_i$, i.e. its i-th component encodes how much the vector $\vec{N}$ would change, were we to perform the i-th Weyl reflection on it. A useful observation is that, by (2.2), we can write $\vec{N}^2 = -N_i \cdot \Delta N_i$, where $\cdot$ is the ordinary inner product in $\mathbb{R}^n$ (no Cartan matrix involved).

We are now ready to discuss the difference between $\vec{N}^2 > 0$ and $\vec{N}^2 < 0$ flows. Given a Weyl orbit, there is a minimum distance — in the canonical euclidean metric of $\mathbb{R}^n$ — from the points representing theories in that orbit to the origin of $\mathbb{R}^n$. If a theory corresponding to such minimum has all N's positive, then for any Weyl reflection acting on it, $N_i' > N_i$, i.e. $\Delta\vec{N}$ has all components positive, and from the observation in the previous paragraph, we deduce that it must correspond to a $\vec{N}^2 < 0$ flow. We then conclude that if all the $N's$ are positive at a minimum, it must be an orbit with $\vec{N}^2 < 0$, or equivalently, all the $\vec{N}^2 > 0$ orbits have $N's$ of mixed signs at the minimum.

Physically, we interpret this difference as follows. Recall that as we flow towards the IR, the N's decrease. On the $\vec{N}^2 > 0$ region, since the minimum of the orbit has $N's$ of mixed signs, as we flow towards the IR, the flow eventually will enter the regions where some of the ranks of the gauge groups would be negative, so they make no sense anymore, exactly as in the KS solution. As in the KS cascade, the IR endpoint for $\vec{N}^2 > 0$ flows is reached when the strongly coupled gauge group has $N_f(i) < N_c(i)$. Again, as in KS, the physics of the IR endpoint will depend on the rest of the N's (what was called p in the previous section), but for endpoints very near the axis of $\mathbb{Z}^n$ we expect again confinement for some range of energy scales (for theories on the axis, pure $\mathrm{SU}(N)$, we expect confinement at all scales).

For the theories in the $\vec{N}^2 < 0$ region, there will be some flows with the behavior just described, but the typical flow in this region will have a different IR behavior. Now the IR endpoint is characterized by $N_c(i)' > N_c(i)$ for all i. In the example we will consider next, we find that IR endpoint can be a non-trivial CFT, or IR free theories.

4.1 An example

Let's illustrate these points with a simple example. Consider $\mathcal{N} = 1$ $d = 4$ gauge theories with gauge group $\mathrm{SU}(N_1) \times \mathrm{SU}(N_2)$ with $k > 2$ chiral multiplets A_i in $(N_1, \bar{N}_2)$, k chirals B_i in $(N_2, \bar{N}_1)$ and quartic superpotential

$$W = \lambda \epsilon^{i_1 i_2 i_3 \ldots i_k} \epsilon_{j_1 j_2 i_3 \ldots i_k} \operatorname{Tr} A_{i_1} B^{j_1} A_{i_2} B^{j_2} \tag{4.1}$$

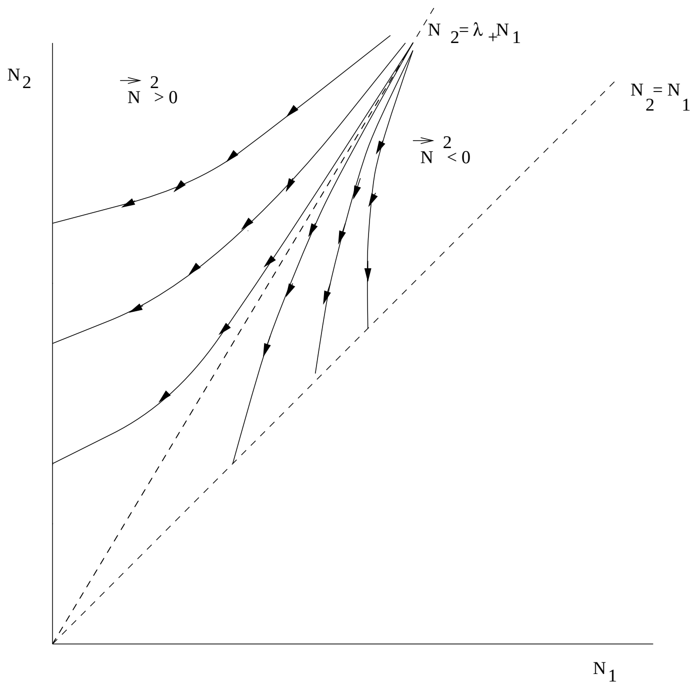

Figure 3: Hyperbolic flow for theories $\mathrm{SU}(N_1) \times \mathrm{SU}(N_2)$ with $k > 2$ hypermultiplets. After modding out by the $\mathbb{Z}_2$ symmetry, the $N_2 = \lambda_+ N_1$ represents the "light-cone" separating the $\vec{N}^2 > 0$ and the $\vec{N}^2 < 0$ regions.

with a $\mathrm{U}(k)$ global continuous symmetry. The Cartan matrix is

$$C = \begin{pmatrix} 2 & -k \\ -k & 2 \end{pmatrix}$$

with eigenvectors $(1,-1)$ and $(1,1)$ (the "time direction"). The Weyl reflections, acting on the right on the row vector $(N_1 \; N_2)$, are

$$R_1 = \begin{pmatrix} -1 & 0 \\ k & 1 \end{pmatrix} \qquad R_2 = \begin{pmatrix} 1 & k \\ 0 & -1 \end{pmatrix} .$$

The $\vec{N}^2 = 0$ cone can be described by defining $\lambda = N_2/N_1$, and rewriting the previous equation as $\lambda^2 - k\lambda + 1 = 0$, which has the solutions

$$\lambda_\pm = \frac{k \pm \sqrt{k^2 - 4}}{2}$$

Note that $\lambda_- = 1/\lambda_+$, which corresponds to the exchange of N_1 and N_2. The straight lines $N_2 = \lambda_\pm N_1$ give the $\vec{N}^2 = 0$ cone in $\mathbb{R}^2$, and the generic hyperbola $\vec{N}^2 = 2(N_1^2 + N_2^2 - kN_1N_2) = s^2$ asymptotes towards this cone. In this example, all the theories with $\vec{N}^2 < 0$ have both N's positive, and correspond to sensible gauge theories. We eliminate the $\mathbb{Z}_2$ symmetry by always considering theories with $N_1 < N_2$.

Since primitive Weyl reflections square to the identity, if we want to have a RG cascade relating an infinite number of theories, we should ensure that we are in a regime where the couplings run in such a way as to require alternate dualizations, as was the case in the

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KS duality cascade. Namely, we would like to determine under which conditions the beta functions for the two groups have different signs. As in [6], we will use the beta functions for inverse square couplings, with holomorphic gauge kinetic term and canonical matter kinetic term,[2]

$$\beta_1\left(\frac{8\pi^2}{g_1^2}\right) = 3N_1 - kN_2(1-\gamma) \qquad \beta_2\left(\frac{8\pi^2}{g_2^2}\right) = 3N_2 - kN_1(1-\gamma) \tag{4.2}$$

with γ the anomalous dimension of the $\mathrm{Tr}(A_iB_j)$ operators, common to all of them due to flavor symmetry. Here a difference appears, compared with the KS solution reviewed in the previous section. In KS it was crucial to take into account the effect of the anomalous dimension γ of $\mathrm{Tr}(A_iB_j)$ on the beta functions of the two gauge couplings, to obtain beta functions with different sign. In the present theories, before taking into account the presence of anomalous dimensions, it is straightforward to find a range of $N_{1,2}$ such that the beta functions for the gauge couplings have different signs. Indeed, as we will show shortly, all the effective theories appearing at different energy scales of the cascade — except the last one in the IR — have two gauge factors that are not in their respective conformal windows, but rather one is magnetically free in the IR, and the other electrically free. The concern is then that the anomalous dimensions don't spoil this property.

In general, we can't compute the anomalous dimensions that appear in the beta functions for the different gauge couplings. For the theories appearing in the KS solution, the strategy was the following: first consider the theories that flow to a nontrivial CFT in the IR (M=0, no fractional branes). For these theories, using the arguments of [25], it is possible to determine the anomalous dimension at the conformal point, $\gamma(Q\bar{Q}) = -1/2$ [2]. Then, for the theories $\mathrm{SU}(N+M)\times\mathrm{SU}(N)$, it was argued that the anomalous dimensions are given by a power series in $(M/N)^2$, $\gamma(Q\bar{Q}) = -1/2 + \mathcal{O}(M^2/N^2)$.[3]

Although we are not able to compute the anomalous dimensions $\gamma(g_1, g_2, \lambda)$, for the arguments we will presenting, it suffices to bound their value. To bound the value of the anomalous dimensions from below, we argue as follows.[4] First recall that for $\mathcal{N}=1$ $\mathrm{SU}(N_c)$ with N_f flavors in the conformal window, the anomalous dimension of the meson at the IR fixed point is given by $\gamma(g_*) = 1 - 3N_c/N_f$ [35], so in this window, $-1 \leq \gamma(g_*) \leq 0$. At the lower end of the conformal window, $N_f = 3N_c/2$, the dimension of the meson $D(\bar{Q}Q)$, reaches its minimum value allowed by unitarity $D(\bar{Q}Q) = 1$, so it becomes free. It is then expected that for $N_f \leq 3N_c/2$, $\gamma_{\mathrm{IR}}(\bar{Q}Q) = -1$. In the case we are studying, the gauge factor that is getting strongly coupled towards the IR is always magnetically free, and although in addition we now have another gauge factor and the quartic superpotential (4.1), we are going to assume that in the strong coupling regime $\gamma \geq -1$.

To bound the anomalous dimensions from above, note that if the quartic superpotential (4.1) becomes marginal, then $\gamma(\bar{Q}Q) = -1/2$ in the IR. If $\gamma(\bar{Q}Q) > -1/2$, the superpotential is an irrelevant operator, and in the IR we would have the original Seiberg

[2]In footnote 3 of [8] it is pointed out that in the KS cascade, this choice is the natural one as dictated by supergravity.

[3]In subsequent work [7], it was argued that for $M \ll N$ and large 't Hooft couplings, the leading correction to the anomalous dimension takes the form $(M/N)^4(1/g_1^2N + 1/g_2^2N)^{1/2}$.

[4]I would like to thank M. Strassler for helpful discussions on this point.

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duality, with the unwanted massless meson. The consistency of the analysis thus requires $\gamma(\bar{Q}Q) \leq -1/2$. We expect the superpotential to become marginal for the theories with group $\mathrm{SU}(N) \times \mathrm{SU}(kN/2)$, since for these theories, the strongly coupled gauge factor, $\mathrm{SU}(kN/2)$, has $N_f = 2N_c$, and if we apply Seiberg duality, the full theory goes back to itself, which we interpret as evidence of a non-trivial CFT in the IR. Then, in the IR the quartic superpotential becomes a marginal operator and $\gamma(\bar{Q}Q) = -1/2$. Since for $N_2/N_1 > k/2$ the theory is "more strongly coupled" in the IR, by analogy with the $\mathrm{SU}(N_c)$ case reviewed above, we expect the anomalous dimensions in the strong coupling to be smaller than $-1/2$. Therefore, for theories with $N_2/N_1 > k/2$, we are going to assume that at the scale we perform the duality, $\gamma \leq -1/2$. Luckily, as we will shortly show, all the flow (except the very last duality transformation in the IR) takes place among theories with $N_2/N_1 > k/2$, so this bound suffices for our purposes. It is worth pointing out that most of the arguments we are going to present would carry through with a more relaxed version of the bounds, $-1 \leq \gamma \leq 0$, so we are confident that the results we present are robust.

Our starting point is then a theory with (N_1, N_2) with, say, $N_2 > N_1$. For reasons that will become clear below, we further require $N_2/N_1 > k-1$; this condition is enough to guarantee that $\beta_2 > 0 > \beta_1$. Since $k - 1 < \lambda_+$, that theory can be either in the $\vec{N}^2 > 0$ or the $\vec{N}^2 < 0$ region. Also, $\beta_2 > 0 > \beta_1$ and the positivity of the couplings require that if we are at an energy scale μ, the respective scales satisfy $\Lambda_2 < \mu < \Lambda_1$. Having established the setup, we are going to study the behavior of such theory as we flow towards the IR. Later we will reconstruct a flow backwards, towards the UV.

As we flow to the IR, and let the couplings run using (4.2), the gauge coupling of the larger group, $\mathrm{SU}(N_2)$, gets stronger, while the gauge coupling of the smaller gauge group gets weaker. Eventually, we reach a scale where $1/g_2^2 = 0$ and we dualize the larger group (i.e., we apply R_2), so $N_1' = N_1$ and $N_2' = kN_1 - N_2$. The new scales are

$$\Lambda_1' = \Lambda_2 \left(\frac{\Lambda_1}{\Lambda_2}\right)^{\beta_1/\beta_1'} \qquad \Lambda_2' = \Lambda_2 \,.$$

Concerning the new β functions for the inverse couplings, it is easy to argue that from $\gamma' \leq -1/2$, it follows that $\beta_2' < 0$, so the dualized gauge group now runs towards weak coupling. On the other hand, the behavior of the spectator gauge group needs a separate discussion.

For theories with $\vec{N}^2 > 0$, just assuming $\gamma' \geq -1$ suffices to show that $\beta_1' > 0$, so the roles of the two gauge groups get exchanged. This scenario is qualitatively similar to the KS cascade. As we keep dualizing the bigger gauge factor at each step, eventually we reach a point where a further transformation would yield $N_2' < 0$, so we have reached the IR endpoint. This happens for $N_2 > kN_1$, and as in KS, for theories near the axis, we expect that the strongly coupled gauge factor presents confinement at some energy scales.

For the $\vec{N}^2 < 0$ theories, as we apply R_2 three things can happen, depending on the ratio N_2/N_1:

i) $N_2' < N_1$, which happens for $\lambda_+ > N_2/N_1 > k-1$,

ii) $N_2 > N_2' > N_1$, which happens for $k - 1 > N_2/N_1 > k/2$,

iii) $N_2' > N_2$, which happens for $k/2 > N_2/N_1 > 1$.

In the first case, we go to a theory with smaller ratio (even after the $\mathbb{Z}_2$ reflection), so we move away from the $\vec{N}^2 = 0$ cone. Again, assuming $\gamma' \geq -1$, we can conclude that $\beta_1' > 0$ and new couplings still run in opposite directions. However, we are no longer guaranteed that the new theory still is in region i). Indeed, one can check analitycally — and it is obvious from figure 3 — , that the slopes get smaller at a faster rate as the N's decrease, so they eventually reach a theory in region ii).

Once that happens, the next point is already in region iii). Note that here $N_{1,2}' > N_{1,2}$. This means that both reflections would mean flowing upwards (in different trajectories, with the same s^2). This is the IR endpoint of the flow. As we argued before, in this range we expect the superpotential to become an irrelevant operator, so we can forget about it in the analysis of the IR endpoint. We have then two different scenarios: for $k/2 \geq N_2/N_1 \geq k/3$, the theory flows to an interacting CFT, with the spectator group playing the role of a weakly gauged flavor symmetry; for $k/3 > N_2/N_1 > 1$ both groups are IR electrically free.

The upshot of this discussion is that for flows with $\vec{N}^2 < 0$, all the effective theories in the cascade, except the last two, satisfy $\lambda_+ > N_2/N_1 > k-1$. Then, near the end of the flow there is a theory with $k-1 > N_2/N_1 > k/2$ and finally a theory with slope $k/2 > N_2/N_1 > 1$, which describes the IR endpoint of the flow.

Having described the flow to the IR, now we would like to study the UV behavior of these flows. What is meant by this is that we let the couplings run towards the UV using (4.2). Eventually one of the couplings will get strong, signalling that the corresponding gauge factor is not asymptotically free, and we propose to produce a new effective theory, valid at higher energy scales, by applying Seiberg duality on that node.

Again, start with a theory with $N_2/N_1 > k-1$, and $\beta_2 > 0 > \beta_1$. Now, letting the couplings run upstream according to (4.2), eventually $1/g_1^2 = 0$, and we apply Seiberg duality to the first gauge factor (i.e., R_1) to obtain the new gauge groups and beta functions to keep flowing towards the UV. It is easy to argue that as we flow back to the UV, the gauge couplings take turns in satisfying $1/g_i^2 = 0$, so we have to apply R_1 and R_2 alternatively. Indeed, after applying R_1, the new gauge theory has $N_1' = kN_2 - N_1$, $N_2' = N_2$ and since we assume $-1 \leq \gamma'$, it follows that after the reflection we have $\beta_1' > 0 > \beta_2'$.

The ranks of the gauge groups form then a series $N_1, N_2, \ldots N_i, \ldots$, so at each step of the cascade we have gauge group $\mathrm{U}(N_n) \times \mathrm{U}(N_{n+1})$. The generic term of the series is

$$N_{n+2} = \frac{\lambda_+^{n+1} - \lambda_-^{n+1}}{\sqrt{k^2-4}} N_2 - \frac{\lambda_+^{n} - \lambda_-^{n}}{\sqrt{k^2-4}} N_1 \,.$$

After many steps up in the cascade, this can be approximated by $N_{n+2} \sim c_1 \lambda_+^n$, so the ranks of the gauge groups grow *exponentially* with n, the number of steps in the cascade. This is to be contrasted with the linear growth of the KS cascade and the affine cascades in general. As a check, from this expression it follows that

$$N_{n+2} - \lambda_+ N_{n+1} \to 0 \qquad \text{as} \qquad n \to \infty$$

independently of N_1 and N_2, so all the flows asymptote in the UV to the straight line $N_2 = \lambda_+ N_1$.

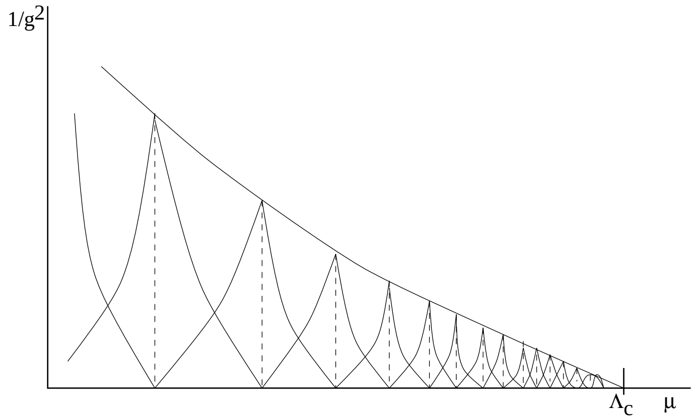

Figure 4: Accumulation of scales for the hyperbolic flow. Towards the UV, the duality cascade can't be extended indefinitely, reaching a "duality wall" at a finite scale Λ_c.

Now we would like to study the behavior of the scales at which one applies the Weyl reflection. On this score we will encounter a surprise. We apply R_1 at the energy scale when $1/g_1^2 = 0$, i.e. $\mu = \Lambda_1$. Matching the scales, we obtain

$$\Lambda_1' = \Lambda_1 \qquad \Lambda_2' = \Lambda_1 \left(\frac{\Lambda_2}{\Lambda_1}\right)^{\beta_2/\beta_2'}$$

Running the couplings with the β functions (4.2), now towards the UV eventually we reach $1/g_2^2 = 0$, and then apply R_2. The net effect of these two reflections is

$$\frac{\Lambda_1''}{\Lambda_2''} = \left(\frac{\Lambda_1}{\Lambda_2}\right)^{\frac{\beta_1'}{\beta_1''}\frac{\beta_2}{\beta_2'}} .$$

As mentioned, now we are guaranteed that the β functions run with opposite signs, so this procedure can be iterated indefinitely. There are a couple of things to notice: first, as we flow upwards, towards the UV, the couplings are getting stronger, contrary to the KS solution, where the couplings were periodic. Second, the scales at which we perform the duality also get closer among themselves. Indeed, we can bound the exponent that appears in the transformation

$$\left|\frac{\beta_s}{\beta_s'}\right| < \frac{2\lambda_+ - k}{(k^2-2)\lambda_+ - k} \equiv r < 1 ,$$

where β_s denotes the beta function of the spectator gauge group at each Seiberg duality, and we have used that $N_2/N_1 \to \lambda_+$. It follows that after n steps

$$\Lambda_n < \Lambda_1 \left(\frac{\Lambda_1}{\Lambda_2}\right)^{\frac{r-r^{n+1}}{1-r}}$$

and as we keep dualizing there is a finite scale that we can't go beyond of

$$\Lambda_c \sim \Lambda_1 \left(\frac{\Lambda_1}{\Lambda_2}\right)^{\frac{r}{1-r}} .$$

The scales pile up towards an accumulation point, defined by a single scale Λ_c. This is quite different from the UV behavior of the KS cascade, which in principle could be extended to arbitrarily high energies. This behavior — mentioned at the end of [6] — was observed by Strassler in some models, and he dubbed it as a "duality wall" [15]. We will discuss the possible meaning of that limiting scale in the next section.

Since both gauge couplings are getting strong, pretending that one of the gauge groups is a spectator and applying Seiberg duality to the other is less and less well motivated as we flow back to the UV. Naively we could try to appeal to holomorphicity in Λ_1/Λ_2: since Seiberg duality applies when $\Lambda_1/\Lambda_2 \ll 1$, we might expect that still holds as we let $\Lambda_1/\Lambda_2 \sim 1$. The KS supergravity solution offers an example where such argument holds. On the field theory side, one of the two gauge couplings ought to be very weak in order that we can confidently apply Seiberg duality to the other factor. On the other hand, the supergravity solution is reliable when $g_s M \gg 1$. Since $1/g_1^2 + 1/g_2^2 = 1/g_s$, in the supergravity description of the cascade, the gauge couplings can't be arbitrarily weak, but the supergravity solution indicates that in that regime, the "continuation" of Seiberg duality still holds.

Additional examples of the validity of such continuation are the duality cascades, both chiral and not chiral, studied in [11]. There, Seiberg duality is realized in the D-brane setup as monodromy in the Teichmüller space of the string theory background. Again, in those examples, from the field theory point of view, naive application of Seiberg duality is not fully justified once all couplings get strong, but thanks to the string theory realization, one is confident that indeed the theories form part of a duality cascade.

4.2 A UV completion?[5]

Given the observations that different flows converge in the UV, and that the energy scales pile up at a finite energy scale Λ_c, it is tempting to speculate that for every k there exist a new theory, which serves as a UV completion of the flows studied in this section. Borrowing language from CFT, this theory would be a "fixed point" (not in the ordinary sense, since it has a mass scale), that could be perturbed by a relevant operator — with coefficient depending on s^2 — triggering the different RG flows. For these flows, at different scales the effective descriptions would be given by the $\mathrm{SU}(N_1) \times \mathrm{SU}(N_2)$ theories.

Throughout this section, the examples we will have in mind are those of $4+1$ and $5+1$ SYM with sixteen supercharges [36]. For each of these two effective theories, a possible UV completion is provided in the context of D-branes in string theory, although ultimately these completions don't involve gravitational physics. These UV completions are respectively given by a 6-dimensional CFT — the $(2,0)$ theory — compactified on a circle, and a 6-dimensional non-local non-gravitational field theory, the $(1,1)$ little string theory. Of course, on top of these scenarios — a CFT or some kind of little string theory — we can't rule out the possibility that simply there is no UV completion of the previous RG cascade. We are going to assume that such UV completion exists, and try to guess some of their properties.

[5] I would like to thank O. Aharony and M. Berkooz for helpful comments on a preliminary version of the following discussion.

First, all the effective theories appearing in a given flow have a continuous U(k) global symmetry, times a discrete symmetry that, for fixed k, depends on the particular flow (e.g. for KS, the discrete global symmetry is $\mathbb{Z}_{2(N_1-N_2)}$). The natural guess would be that the theory providing the UV completion has a global U(1) which is broken to discrete subgroups when we turn the relevant operator that triggers the flow. Thus, the global symmetry group ought to be U(k) $\times$ U(1). Also, all the effective theories that enter the cascade have 4 supercharges. If the UV completion is a higher dimensional ($d = 5, 6$) CFT, it has at least 8 supercharges, so the relevant operator that triggers the cascade ought to break some supersymmetry.

What would be the asymptotic density of states of these hypothetical theories? It is straightforward to derive the growth of the ranks of the gauge groups as we approach Λ_c; near Λ_c we have SU(N) $\times$ SU($\lambda_+ N$) with

$$N \sim \left(\frac{\Lambda_c}{\Lambda_c - \Lambda}\right)^{\frac{\ln \lambda_+}{|\ln r|}} .$$

The hard question is, of course, to estimate the number of degrees of freedom of the quiver theories when both couplings are strong. We could wonder whether the number of degrees of freedom remains constant as we approach Λ_c, but this seems to be against the spirit of dualizing to improve the UV behavior of a not asymptotically free theory, so the possibility that the number of degrees of freedom diverges as we approach Λ_c seems more in consonance with the picture developed so far. If this is indeed the case, this growth of the number of degrees of freedom seems to rule out the higher dimensional CFT scenario, which would imply a finite number of degrees of freedom at any energy scale. On the other hand, this behavior has some similarity with that of $6d$ little string theories above their mass gap $M^2 = M_s^2/N$, where a continuum of states develops [37, 38, 39], giving (admittedly mild) support to the idea that the UV completion ought to be a LST rather than a CFT.

In order to decide the existence of such theories and to explore its properties, it would be important to find a dual realization of them, in string theory. The first question to address is if the dual of this RG flow can have a supergravity description in some regime. For a type II supergravity solution to be reliable, it should have small curvature and small dilaton; if the dilaton is large, a dual description (in 11d SUGRA, or a S-dual solution) would be called for. The inverse curvature in string units is typically given in gauge variables by $\lambda = g_{\mathrm{YM}}^2 N$, so near the $\mu \sim \Lambda_c$ scale, a supergravity dual could have a small curvature. Near the IR endpoint, we expect a SUGRA dual description to exist only in the absence of weakly coupled degrees of freedom on the gauge theory side, as in the KS cascade.

Concerning the dilaton, the best we can do is to recall some cases of gravity duals involving non-asymptotically free theories, and hope that they capture generic features, also present in our duality cascades. If we consider the D-brane description of 4+1 and 5+1 SYM with sixteen supercharges [36], in both cases at low energies we start with a description where the effective gauge theory is useful, but being non-renormalizable, eventually breaks down, being substituted by a D-brane supergravity description. As the

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energy scale increases, there is a competing effect on the size of the dilaton, between the effective gauge coupling and N [36], and in both cases for large enough energy scale the dilaton grows large and we have to resort to a dual description.

For the present duality cascade, the ranks N grow as $N \sim \lambda_+^n$, with n the number of steps in the cascade, and in the strong coupling regime, the coupling grows as $g_i^2 \sim (1/r)^n$. Although we have no means to determine which particular combination of gauge couplings and N should enter the expression of the dilaton for a potential dual background, given that $1/r$ is larger than λ_+, the previous analogy would suggest that eventually such dilaton would grow large, requiring a dualization of the supergravity description.

In short, for most of the flows, we don't expect a supergravity description for the extreme IR of the hyperbolic duality cascade. If we imagine that these gauge theories appear in the worldvolume of some brane configuration, it is possible that in some regime near $\mu \sim \Lambda_c$ a supergravity description is valid, possibly involving a supergravity configuration dual to the original one (i.e, in 11d supergravity or the S-dual). The duality wall might then be realized in supergravity in a similar fashion as the mass gap of the LST's is realized in their supergravity duals.

Finally, we could ask how the difference between $\vec{N}^2 > 0$ and $\vec{N}^2 < 0$ flows can show up in the dual. If we assume that some form of the UV/IR relation holds for the present cascade, since both kinds of flows have the same UV behavior, we expect them to correspond to string backgrounds with the same asymptotic behavior. On the other hand, the IR behaviors are quite different, but since we don't expect a supergravity description to be valid in that regime, it is not clear how this difference would manifest itself in a dual description.

Needless to say, the present discussion is quite speculative in nature. We are suggesting the existence of a theory which — given the supposed grotwh of the number of degrees of freedom — is not a QFT, serving as a possible UV completion for the hyperbolic duality cascades discussed in this section. One could try to turn this suggestion around, and look for 4d little string theories (maybe along the lines of [40]) with a global $\mathrm{U}(k) \times \mathrm{U}(1)$ symmetry, and try to relate them with the duality cascades just considered.

5. Conclusions

In this work, extending some ideas of [11], we have recasted the phenomenon of duality cascade of [6] in terms of the Cartan matrix underlying the quiver gauge theories that appear in the cascade, and we have generalized it to other duality cascades. The Weyl reflections of the Cartan matrix correspond to Seiberg dualities, and the UV behavior of these cascades depends on the kind of Cartan matrix.

One could consider a number of generalizations to the present work. In [41, 42, 43] duality cascades involving products of SO and Sp gauge groups are investigated, and the formalism presented here could be applied to them. There have been some discussions on duality for non supersymmetric quiver theories [44, 45, 46, 47]. It is clearly important to further study these matters, and to explore the possibility of non-supersymmetric RG cascades.

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We studied in some detail a simple example of gauge theories with hyperbolic Cartan matrix, and found some evidence for the existence of some new theory with a fixed scale Λ and global symmetry $\mathrm{U}(k)$, which properties somewhat similar to those of little string theories. Establishing the existence of these theories, and realizing them by some independent construction, would be quite interesting.

More generally, it is not known which not asymptotically free (NAF) theories admit a UV completion in terms of an ordinary QFT. In 4 dimensions, $\mathcal{N} = 4$ gauge theories are always controlled in the UV by the CFT at the origin of moduli space, so the question is trivially answered. For $\mathcal{N} = 2$ gauge theories, some results are known: $\mathrm{SU}(N_c)$ with N_f flavors is NAF for $N_f \geq 2N_c$, but one can argue [48] that all these theories appear as relevant perturbations of ordinary CFT's (although for N_f odd, without a lagrangian description). Similarly, $\mathcal{N} = 2$ quiver theories with gauge group $\prod \mathrm{SU}(N_i)$ and ADE or affine ADE Cartan matrix can be obtained by higgsing of a quiver CFT with the same Cartan matrix. On the other hand, I am not aware of a straightforward UV completion for $\mathcal{N} = 2$ quiver theories with more general Cartan matrices, e.g., with hyperbolic Cartan matrices. Although deciding the existence or not of such UV completions seems like a hard problem, one might hope that progress in this direction would lead towards a new way to deduce the existence of new non-gravitational theories, without taking decoupling limits from string theory.

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